



Article Type: Original Research Article

Exact Asymptotically Flat Traversable Wormholes in Rastall Teleparallel Gravity

Khilola Khudarganova^{1*}, Shakhlo Jabbarova¹

¹Institute of Fundamental and Applied Research, National Research University TIIAME, Kori Niyoziy 39, Tashkent 100000, Uzbekistan.

*Corresponding Author: hxudarganova@gmail.com

Received: 07 August 2026 **Accepted:** 03 October 2026 **Published:** 12 October 2026

Abstract

We construct static, spherically symmetric traversable wormholes in the teleparallel version of Rastall gravity, in which the divergence of the matter energy-momentum tensor is tied to the gradient of the torsion scalar, $\nabla_\mu \Theta^\mu = \lambda \partial_\mu T$. Working with the non-diagonal “good” tetrad, zero tidal force and a linear radial equation of state $p_r = \omega \rho$, the field equations integrate exactly. The shape function is obtained in closed parametric form; it reduces to the known power-law wormhole of general relativity when the Rastall parameter $\zeta \rightarrow 0$ and remains asymptotically flat for $|\zeta| < 1/2$. By contrast, the diagonal tetrad yields a geometry with a solid-angle deficit, which gives an independent argument for the non-diagonal frame. Three exact results organize the physics: (i) the radial null energy condition is insensitive to the Rastall term pointwise and is violated at the throat, as flaring out demands; (ii) the strong-energy combination reduces identically to $\kappa(\rho + p_r + 2p_t) = -2\zeta T$, so it holds everywhere for $\zeta < 0$; and (iii) the volume-integral quantifier takes the compact form $I_V = -2 \int_r^\infty s \rho ds$ with $s = \sqrt{1 - b/r}$, and decreases in magnitude as ζ becomes more negative. Embedding diagrams, proper distance, anisotropy, hydrostatic equilibrium through a Rastall-modified Tolman–Oppenheimer–Volkoff equation and tidal constraints are analysed. Negative coupling reduces the exotic-matter budget near the throat and restores the strong energy condition, at the cost of the tangential dominant energy condition close to the throat for $\zeta \lesssim -0.1$.

Keywords: Traversable wormholes; Rastall gravity; teleparallel gravity; torsion scalar; energy conditions; exotic matter

1. Introduction

The idea of a spacetime bridge goes back to Flamm’s reading of the Schwarzschild geometry [1] and to the Einstein–Rosen construction [2], later recast by Misner and Wheeler in geometrodynamical language [3]. The modern programme started when Morris and Thorne asked which static geometries a traveller could cross in finite proper time without meeting a horizon or destructive tidal forces [4, 5]. Their answer was a set of conditions on the shape and redshift functions that, through the Einstein equations, forces

the matter at the throat to violate the null energy condition (NEC) [6, 7]. Scalar-field throats [8, 9], phantom-energy sources [10, 11], thin shells [12, 13] and quantifiers that measure how much exotic matter is needed [14] grew out of this observation.

A second line of work moved the problem into modified gravity, where additional geometric terms can carry part of the burden of the flare-out condition. Wormholes have been studied in $f(R)$ gravity [15], in theories with curvature–matter coupling [16, 17], in hybrid metric–Palatini gravity [18], in Einstein–Gauss–Bonnet type theories with dilaton [19] and in astrophysical settings such as galactic halos [?]. Teleparallel gravity offers a different starting point. There the gravitational field is carried by a tetrad and the curvature-free Weitzenböck connection, and the teleparallel equivalent of general relativity (TEGR) differs from the Einstein–Hilbert action only by a boundary term [21, 22, 23]. Its $f(T)$ extensions [24, 25, 26, 27] have been used for wormholes by Böhmer, Harko and Lobo [28] and by Jamil, Momeni and Myrzakulov [29], among others. A recurring lesson of that literature is that once local Lorentz invariance is lost, the choice of tetrad matters, and the non-diagonal “good” tetrads of Refs.[30, 31, 32] are preferred to the naive diagonal frame [33, 34]

Rastall proposed a different relaxation of general relativity (GR): the energy–momentum tensor need not be covariantly conserved in curved spacetime, and its divergence may be proportional to the gradient of the Ricci scalar [35, 36]. The resulting field equations contain a single extra parameter, and their vacuum sector coincides with GR. The status of the theory has been debated. Visser argued that Rastall gravity is a rearrangement of the matter sector of GR [37], while Darabi *et al.* replied that the two theories are inequivalent once the matter content is fixed by physics rather than by bookkeeping [38, 39]; see also Ref.[40]. Independent of that discussion, Rastall gravity has been used to model neutron stars [41], black holes [42, 43, 44], scalar-field configurations [45], compact stars [46] and wormholes [47, 48, 49].

Saaidi and Nazavari combined the two ideas by replacing the Ricci scalar in Rastall’s postulate with a function of the torsion scalar [50]. The resulting Rastall teleparallel framework has since been applied to wormholes with a non-diagonal tetrad [51], to non-commutative wormhole sources [52], and to anisotropic and dark-energy compact stars [53, 54, 55]. Teleparallel stellar models with non-linear equations of state, built on the same field equations used here, were analysed in Ref.[56].

What is missing is a wormhole family in the linear theory ($f(T) = T$, $h(T) = T$) that is (a) exact, (b) asymptotically flat, (c) continuously connected to the GR solution, and (d) explicit enough that the role of the Rastall coupling can be assessed.

pling can be isolated analytically rather than read off from plots.

This paper provides such a family. With zero tidal force and the linear equation of state $p_r = \omega\rho$, the Rastall teleparallel equations reduce to a separable first-order equation whose solution we give in closed parametric form. Sec. II reviews the framework, Sec. III derives the wormhole equations for both tetrads, Sec. IV presents the exact solution, Sec. V analyses its physical properties, and Sec. VI summarizes. We use $G = c = 1$, signature $(+, -, -, -)$ and $\kappa = 8\pi$.

2. Rastall teleparallel gravity

Spacetime indices are denoted by Greek letters and tangent-space indices by Latin letters. The metric is built from the tetrad e^i_μ as [21, 56]

$$g_{\mu\nu} = \eta_{ij} e^i_\mu e^j_\nu, \quad \eta_{ij} = \text{diag}(1, -1, -1, -1), \quad (1)$$

with $e = \det(e^i_\mu) = \sqrt{-g}$. The Weitzenböck connection

$$\Gamma^\alpha_{\mu\nu} = e_i^\alpha \partial_\nu e^i_\mu \quad (2)$$

is curvature-free and has torsion

$$T^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu} - \Gamma^\alpha_{\mu\nu} = e_i^\alpha (\partial_\mu e^i_\nu - \partial_\nu e^i_\mu). \quad (3)$$

The contorsion

$$K^{\alpha\mu}{}_{\nu} = -\frac{1}{2} (T^{\alpha\mu}{}_{\nu} + T^{\mu\alpha}{}_{\nu} - T_{\nu}{}^{\alpha\mu}) \quad (4)$$

relates the Weitzenböck and Levi-Civita connections through

$$\tilde{\Gamma}^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} - K^{\alpha}_{\mu\nu}. \quad (5)$$

With the superpotential

$$S^{\mu\nu}_{\alpha} = \frac{1}{2} \left(K^{\mu\nu}{}_{\alpha} + \delta^{\mu}_{\alpha} T^{\beta\nu}{}_{\beta} - \delta^{\nu}_{\alpha} T^{\beta\mu}{}_{\beta} \right), \quad (6)$$

the torsion scalar is $T = S^{\mu\nu}_{\alpha} T^{\alpha}_{\mu\nu}$. It differs from the Ricci scalar of the Levi-Civita connection by a local divergence,

$$R = -T + B, \quad B = \frac{2}{e} \partial_{\mu} (e T^{\nu\mu}{}_{\nu}), \quad (7)$$

up to convention-dependent signs [?], so the TEGR action

$$S = \int d^4x e \left[\frac{T}{2\kappa} + \mathcal{L}_m \right] \quad (8)$$

reproduces the Einstein equations. Varying with respect to the tetrad gives

$$\begin{aligned} G^{\mu}_{\nu} \equiv 2 \left[e^{-1} e^i{}_{\nu} \partial_{\alpha} (e e^{\rho}{}_i S^{\mu\alpha}_{\rho}) \right. \\ \left. + T^{\rho}_{\alpha\nu} S^{\mu\alpha}_{\rho} + \frac{1}{4} \delta^{\mu}_{\nu} T \right] = \kappa \Theta^{\mu}_{\nu}. \end{aligned} \quad (9)$$

This is Eq. (11) of Ref.[56] written with $\kappa = 8\pi$. The tensor G^{μ}_{ν} equals the Einstein tensor G^{μ}_{ν} identically, for any tetrad of the metric [21, 22]. Consequently, $\nabla_{\mu} \Theta^{\mu\nu} = 0$, where ∇ is the Levi-Civita derivative, which coincides with the teleparallel derivative $D_{\nu} V^{\mu} = \partial_{\nu} V^{\mu} + (\Gamma^{\mu}_{\lambda\nu} - K^{\mu}_{\lambda\nu}) V^{\lambda}$, and the usual conservation law follows.

Rastall's proposal replaces this conservation law by $\nabla_{\mu} \Theta^{\mu\nu} = \lambda \partial^{\nu} R$ [35]. Following Refs. [50, 51], its teleparallel counterpart ties the non-conservation to the torsion scalar,

$$\nabla_{\mu} \Theta^{\mu\nu} = \lambda \partial^{\nu} h(T). \quad (10)$$

We take the linear choice $h(T) = T$, the direct analogue of Rastall's original assumption. Consistency of Eq. (10) with the Bianchi identity requires

$$G^{\nu}_{\mu} + \zeta T \delta^{\nu}_{\mu} = \kappa \Theta^{\nu}_{\mu}, \quad \zeta \equiv \kappa \lambda, \quad (11)$$

so that $\Theta^{\nu}_{\mu} - (\zeta/\kappa) T \delta^{\nu}_{\mu}$ is the conserved combination. The dimensionless Rastall parameter ζ measures the strength of the torsion-matter exchange, and $\zeta = 0$ returns TEGR, i.e. GR. Two remarks are in order. First, since T is not a local Lorentz scalar, the term $\zeta T \delta^{\nu}_{\mu}$ depends on the tetrad, exactly as in $f(T)$ gravity; we return to this in Sec. III. Second, the relation between κ and Newton's constant is renormalized in Rastall-type theories [47, 51]. All the results below are stated for the dimensionless combinations $\kappa r_0^2 \rho$, $\kappa r_0^2 p_{r,t}$, so they do not depend on how that calibration is fixed. We restrict attention to $|\zeta| < 1/2$, and the values used in the figures avoid the points $\zeta = 1/4$ and $1/6$ at which the Newtonian calibration quoted in Ref.[51] becomes singular.

For the matter sector we take an anisotropic fluid,

$$\begin{aligned} \Theta^{\mu}_{\nu} = (\rho + p_t) u_{\mu} u^{\nu} - p_t \delta^{\nu}_{\mu} \\ + (p_r - p_t) v_{\mu} v^{\nu}, \end{aligned} \quad (12)$$

with $u_{\mu} u^{\mu} = -v_{\mu} v^{\mu} = 1$, so that $\Theta^{\mu}_{\nu} = \text{diag}(\rho, -p_r, -p_t, -p_t)$.

3. Wormhole geometry and field equations

We use the Morris–Thorne line element [4]

$$ds^2 = e^{2\Phi(r)} dt^2 - \frac{dr^2}{1 - b(r)/r} - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (13)$$

where Φ is the redshift function and b the shape function. Traversability requires: no horizon (Φ finite); the throat condition $b(r_0) = r_0$ with $b(r) < r$ for $r > r_0$; the flare-out condition $(b - rb')/b^2 > 0$, which reduces to $b'(r_0) < 1$ at the throat; and, for asymptotic flatness, $b/r \rightarrow 0$ as $r \rightarrow \infty$ [4, 6]. We take $\Phi' = 0$, so that radial tidal forces vanish.

Two tetrads reproduce Eq. (13). The diagonal one,

$$e^i_\mu = \text{diag} \left(e^\Phi, (1 - b/r)^{-1/2}, r, r \sin\theta \right), \quad (14)$$

was used in Ref. [56]. The non-diagonal tetrad [30, 31, 57] rotates the spatial triad so that it is aligned with the Cartesian axes at infinity; it is the frame used in the Rastall teleparallel literature [51, 55]. Writing $e^\Phi = (1 - b/r)^{1/2}$ and computing T from Eqs. (3) and (6) (we verified both results symbolically), one finds

$$T_{\text{diag}} = \frac{2e^{-\Phi}}{r} \left(2\Phi' + \frac{1}{r} \right), \quad (15)$$

$$T_{\text{good}} = \frac{2e^{-\Phi}}{r^2} \left(e^{\Phi/2} - 1 \right) \left(e^{\Phi/2} - 1 - 2r\Phi' \right). \quad (16)$$

The first agrees with Eq. (20) of Ref.[56] and the second with Eq. (23) of Ref.[51]. It is convenient to introduce

$$s(r) = \sqrt{1 - \frac{b(r)}{r}}, \quad 0 \leq s < 1. \quad (17)$$

which vanishes at the throat and tends to one at spatial infinity. For $\Phi' = 0$,

$$T_{\text{diag}} = \frac{2s^2}{r^2}, \quad T_{\text{good}} = \frac{2(1-s)^2}{r^2}. \quad (18)$$

The two scalars behave in opposite ways at the throat: T_{diag} vanishes there, while $T_{\text{good}} = 2/r_0^2$ is maximal. With the non-diagonal tetrad the Rastall term is therefore felt most strongly exactly where the geometry flares out. Since $\mathcal{G}_\mu{}^\nu = G_\mu{}^\nu$, Eq. (9) gives, for $\Phi' = 0$,

$$\kappa\rho = \frac{b'}{r^2} + \zeta T, \quad (19)$$

$$\kappa p_r = -\frac{b}{r^3} - \zeta T, \quad (20)$$

$$\kappa p_t = \frac{b - rb'}{2r^3} - \zeta T. \quad (21)$$

These relations already contain three statements that hold for any shape function. The Rastall term is proportional to δ_μ^ν , so it cancels in

$$\kappa(\rho + p_r) = \frac{rb' - b}{r^3}, \quad \kappa(\rho + p_t) = \frac{b + rb'}{2r^3}. \quad (22)$$

The radial NEC is controlled by geometry alone: at the throat $\kappa(\rho + p_r) = (b'(r_0) - 1)/r_0^2 < 0$ by flare-out, for every ζ . Rastall coupling cannot remove the exotic matter at the throat. It can change how much is needed elsewhere, but only through the shape function it selects. Second, because $\rho + p_r + 2p_t$ vanishes identically in GR for $\Phi' = 0$,

$$\kappa(\rho + p_r + 2p_t) = -2\zeta T. \quad (23)$$

With the non-diagonal tetrad $T \geq 0$, so the strong energy condition (SEC) holds everywhere for $\zeta < 0$ and fails everywhere for $\zeta > 0$. Third, the conserved combination of Eq. (9) gives the Rastall-modified Tolman–Oppenheimer–Volkoff (TOV) equation [51]

$$\underbrace{-\frac{dp_r}{dr}}_{F_h} + \underbrace{\frac{2}{r}(p_t - p_r)}_{F_a} - \underbrace{\frac{\zeta}{\kappa} \frac{dT}{dr}}_{F_\zeta} = 0, \quad (24)$$

where the gravitational force $-\Phi'(\rho + p_r)$ is absent for zero tidal force. The torsion gradient acts as an extra force on the fluid.

4. Exact solution

Equations (19)–(21) contain four unknown functions. We close the system with the linear radial equation of state

$$p_r = \omega \rho, \quad (25)$$

widely used for phantom-supported wormholes [10, 11]. In GR it leads to the power law $b = r_0(r/r_0)^{-1/\omega}$. Inserting Eqs. (19) and (20) into Eq. (25) and writing $u \equiv b/r = 1 - s^2$, we find, with the non-diagonal tetrad,

$$r \frac{du}{dr} = -\frac{1}{q} \left[u + 2\zeta(1 - \sqrt{1 - u})^2 \right], \quad q \equiv \frac{\omega}{1 + \omega}. \quad (26)$$

In terms of s the bracket factorizes as $(1 - s)(\mathcal{A} + \mathcal{B}s)$ with

$$\mathcal{A} = 1 + 2\zeta, \quad \mathcal{B} = 1 - 2\zeta, \quad (27)$$

and Eq. (26) becomes separable,

$$\frac{dr}{r} = \frac{2q s ds}{(1 - s)(\mathcal{A} + \mathcal{B}s)} = q \left[\frac{1}{1 - s} - \frac{\mathcal{A}}{\mathcal{A} + \mathcal{B}s} \right] ds. \quad (28)$$

Integrating from the throat ($s = 0, r = r_0$) gives the exact solution in parametric form,

$$\frac{r(s)}{r_0} = (1 - s)^{-q} \left(1 + \frac{\mathcal{B}}{\mathcal{A}} s \right)^{-q\mathcal{A}/\mathcal{B}}, \quad (29)$$

$$b(s) = r(s) (1 - s^2), \quad 0 \leq s < 1. \quad (30)$$

For $\zeta = 0$ one has $\mathcal{A} = \mathcal{B} = 1$, Eq. (29) reduces to $r = r_0(1 - s^2)^{-q}$, and Eq. (30) becomes $b = r_0(r/r_0)^{-1/\omega}$, the GR (TEGR) result. We checked symbolically and numerically that Eqs. (29)–(30) satisfy Eq. (25) and the TOV equation (24) to machine precision.

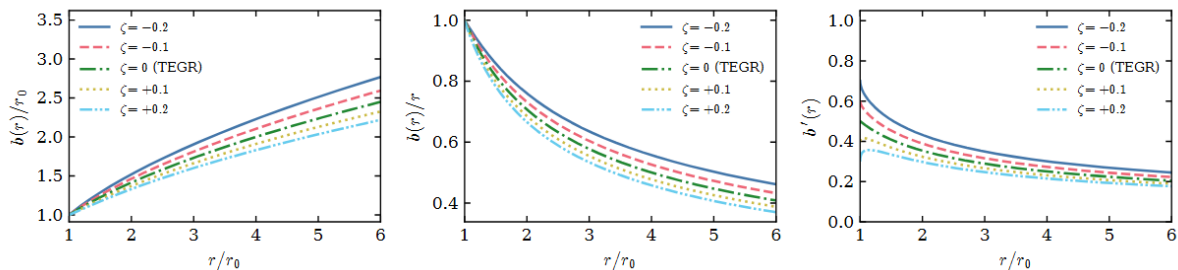


Figure 1: Shape function $b(r)$ (left), ratio b/r (middle) and $b'(r)$ (right) for $\omega = -2$ and the indicated values of the Rastall parameter ζ . Flare-out ($b' < 1$) and asymptotic flatness ($b/r \rightarrow 0$) hold for all curves.

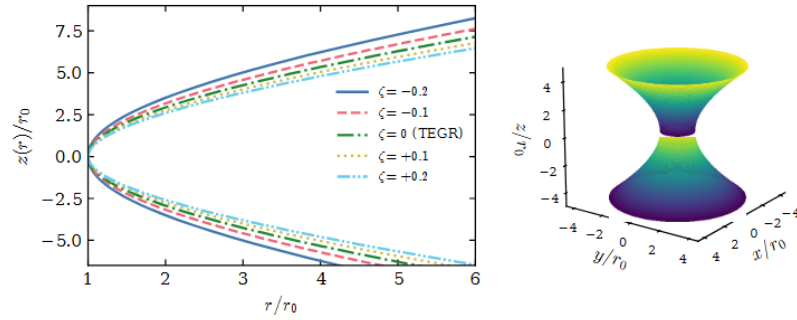


Figure 2: Left: embedding profiles $z(r)$ (both sheets) for $\omega = -2$. Right: embedded surface for $\zeta = 0.2$, $r \leq 4r_0$.

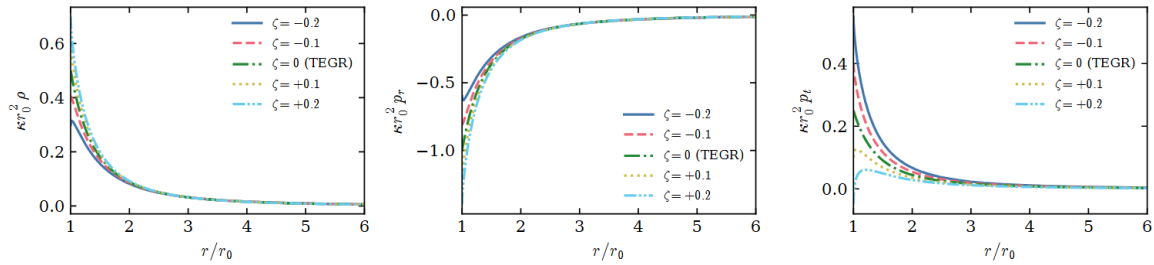


Figure 3: Energy density ρ (left), radial pressure p_r (middle) and tangential pressure p_t (right), in units of $1/(\kappa r_0^2)$, for $\omega = -2$.

Traversability conditions. For $\omega < -1$ (phantom regime) we have $q > 1$, and for $|\zeta| < 1/2$ both $\mathcal{A}$ and $\mathcal{B}$ are positive, so $dr/ds > 0$ on $(0, 1)$ and the map $s \mapsto r$ is one-to-one from $[0, 1)$ onto $[r_0, \infty)$. The throat condition holds by construction and $b < r$ for $r > r_0$ because $s > 0$. Differentiating, $b' = u - (1 - s)(\mathcal{A} + \mathcal{B}s)/q$, hence

$$b'(r_0) = 1 - \frac{(1 + 2\zeta)(1 + \omega)}{\omega} < 1, \quad (31)$$

so the throat flares out for every $\zeta > -1/2$. As $s \rightarrow 1$, Eq. (29) gives $1 - s \simeq (2/\mathcal{A})^{-A/B}(r/r_0)^{-1/q}$ and

$$\frac{b}{r} \simeq 2 \left(\frac{\mathcal{A}}{2} \right)^{A/B} \left(\frac{r}{r_0} \right)^{-(1+\omega)/\omega} \rightarrow 0, \quad (32)$$

so the spacetime is asymptotically flat, with $b \propto r^{-1/\omega}$ as in GR but a ζ -dependent amplitude.

Throat values. From Eqs. (19)–(21),

$$\begin{aligned} \kappa r_0^2 \rho_0 &= -\frac{1 + 2\zeta}{\omega}, & \kappa r_0^2 p_{r0} &= -(1 + 2\zeta), \\ \kappa r_0^2 p_{t0} &= \frac{(1 + 2\zeta)(1 + \omega)}{2\omega} - 2\zeta. \end{aligned} \quad (33)$$

The density at the throat is positive for $\omega < 0$ and $\zeta > -1/2$; the Rastall term rescales it by $1 + 2\zeta$.

Diagonal tetrad. Repeating the steps with T_{diag} gives a linear equation whose solution is

$$b_{\text{diag}}(r) = \frac{r_0}{1 - 2\zeta} \left[\left(\frac{r}{r_0} \right)^m - 2\zeta \frac{r}{r_0} \right], \quad (34)$$

$$m = 1 + \frac{(2\zeta - 1)(1 + \omega)}{\omega}.$$

For $\omega < -1$ one has $m < 1$, and then $b_{\text{diag}}/r \rightarrow -2\zeta/(1 - 2\zeta) \neq 0$. The geometry approaches $ds^2 \simeq dt^2 - d\bar{r}^2 - (1 - k)\bar{r}^2 d\Omega^2$ with $k = -2\zeta/(1 - 2\zeta)$, a solid-angle deficit ($\zeta < 0$) or excess ($\zeta > 0$) of the global-monopole type [58]. Any $\zeta \neq 0$ spoils asymptotic flatness in the diagonal frame, while the non-diagonal frame preserves it. This gives a physical reason, in addition to the formal one [30], to work with the non-diagonal tetrad, and we do so from here on.

5. Physical properties

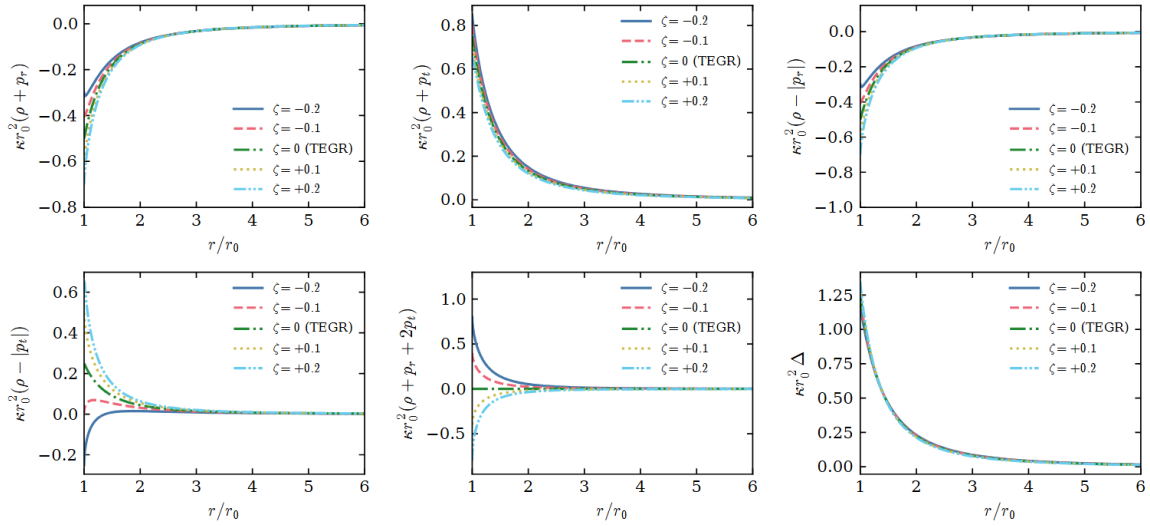


Figure 4: Energy-condition combinations and anisotropy $\Delta = p_t - p_r$, in units of $1/(\kappa r_0^2)$, for $\omega = -2$. The middle-lower panel equals $-2\zeta r_0^2 T$ identically [Eq. (23)].

We fix $\omega = -2$ and take $\zeta \in \{-0.2, -0.1, 0, 0.1, 0.2\}$; the effect of ω is examined at the end of this section. All quantities are shown in units of r_0 .

5.1. Shape function and embedding

Figure 1 shows that all members of the family satisfy the requirements, $b(r)$ grows monotonically from r_0 , b/r decreases from one towards zero, and $b' < 1$ throughout. Negative ζ raises b at fixed r and makes the throat less sharply flared, in agreement with Eq. (31), which gives $b'(r_0) = (1 - 2\zeta)/2$ for $\omega = -2$.

The equatorial slice $t = \text{const}$, $\theta = \pi/2$ embeds in Euclidean space as $z(r)$ with $dz/dr = \pm(r/b - 1)^{-1/2}$. The integrand diverges at the throat, but in the variable s it is regular:

$$\frac{dz}{ds} = \frac{2qr(s)\sqrt{1-s^2}}{(1-s)(\mathcal{A} + \mathcal{B}s)}, \quad \frac{d\ell}{ds} = \frac{2qr(s)}{(1-s)(\mathcal{A} + \mathcal{B}s)}, \quad (35)$$

where ℓ is the proper radial distance, $d\ell = dr/s$. Both are finite everywhere, so ℓ is finite between any two radii and the throat is crossed in finite proper distance. Figure 2 shows the two-sheeted profiles

Table 1: Throat quantities for $\omega = -2$ [Eqs. (31), (33), (23)], the volume-integral quantifier I_V of Eq. (36) for two cut-off radii, and the maximum crossing speed $v_{\max}$ of Eq. (37) for $r_0 = 1$ km. Densities and pressures are in units of $1/(\kappa r_0^2)$.

ζ	-0.2	-0.1	0	0.1	0.2
$b'(r_0)$	0.700	0.600	0.500	0.400	0.300
ρ_0	0.300	0.400	0.500	0.600	0.700
p_{r0}	-0.600	-0.800	-1.000	-1.200	-1.400
p_{t0}	0.550	0.400	0.250	0.100	-0.050
$(\rho + p_t)_0$	0.850	0.800	0.750	0.700	0.650
$(\rho + p_r + 2p_t)_0$	0.80	0.40	0.00	-0.40	-0.80
$I_V(2r_0)/r_0$	-0.346	-0.383	-0.414	-0.442	-0.467
$I_V(5r_0)/r_0$	-1.167	-1.206	-1.236	-1.260	-1.280
$v_{\max} \text{ (m s}^{-1}\text{)}$	5718	4952	4429	4044	3744

Table 2: Status of the energy conditions for $\omega = -2$ on $r \geq r_0$ ($\checkmark$: satisfied everywhere; $\times$: violated everywhere; “= 0”: saturated at the throat and satisfied beyond it).

Condition	-0.2	-0.1	0	0.1	0.2
$\rho \geq 0$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
$\rho + p_r \geq 0$	$\times$	$\times$	$\times$	$\times$	$\times$
$\rho + p_t \geq 0$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
$\rho \geq p_r $	$\times$	$\times$	$\times$	$\times$	$\times$
$\rho \geq p_t $	$r \gtrsim 1.37r_0$	= 0	$\checkmark$	$\checkmark$	$\checkmark$
$\rho + p_r + 2p_t \geq 0$	$\checkmark$	$\checkmark$	= 0 ¹	$\times$	$\times$
$\Delta > 0$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$

and a three-dimensional rendering for $\zeta = 0.2$. Negative ζ opens the funnel more slowly, which is the embedding counterpart of the larger $b(r)$ in Fig. 1; the proper distance to a given r grows accordingly (Fig. 6, right).

5.2. Matter content and energy conditions

The energy density and pressures are plotted in Fig. 3. The density is positive and decreases monotonically outward, with $\rho \propto r^{-3-1/\omega}$ at large r . The radial pressure is negative, as required by $p_r = \omega\rho$ with $\omega < 0$. The tangential pressure is positive except in a thin layer near the throat for the largest positive coupling: Eq. (33) gives $\kappa r_0^2 p_{t0} = (1 + 2\zeta)/4 - 2\zeta$ for $\omega = -2$, which turns negative for $\zeta > 1/6$.

The standard pointwise energy conditions are [6]: NEC, $\rho + p_i \geq 0$; WEC, $\rho \geq 0$ together with the NEC; DEC, $\rho \geq |p_i|$; SEC, $\rho + p_i \geq 0$ and $\rho + p_r + 2p_t \geq 0$, with $i = r, t$. Figure 4 and Table 2 summarize their status. The radial NEC, and therefore the WEC, is violated throughout, with $\kappa(\rho + p_r) = (1 + \omega)\kappa\rho < 0$, in line with the general statement after Eq. (22). The tangential NEC holds everywhere, so the exotic character is confined to the radial direction. The radial DEC fails because $|\omega| > 1$. The tangential DEC is the one condition that discriminates between signs of ζ : for $\zeta = -0.2$ the throat value $\kappa r_0^2(\rho - |p_t|) = -0.25$, and the condition is restored only for $r \gtrsim 1.37r_0$; for $\zeta = -0.1$ it is saturated at the throat and holds beyond it; for $\zeta \geq 0$ it holds everywhere. The SEC behaves in the opposite way, following Eq. (23) exactly: it is satisfied for $\zeta < 0$, marginal in TEGR, and violated for $\zeta > 0$. The anisotropy $\Delta = p_t - p_r$ is positive everywhere (lower-right panel of Fig. 4), so the anisotropic force is repulsive and points outward.

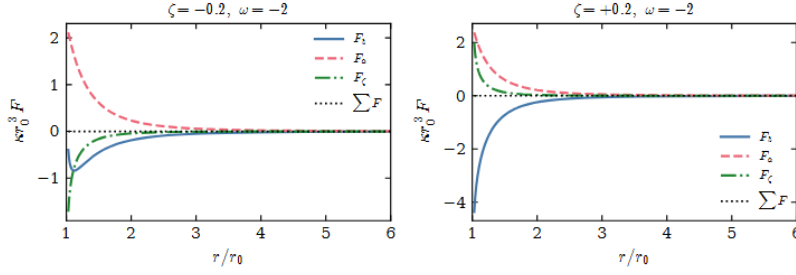


Figure 5: Hydrostatic (F_h), anisotropic (F_a) and Rastall (F_ζ) forces of the modified TOV equation (24), in units of $1/(\kappa r_0^3)$, for $\zeta = -0.2$ (left) and $\zeta = 0.2$ (right); the dotted line is their sum.

5.3. Equilibrium

Figure 5 shows the three forces of Eq. (24) for $\zeta = \mp 0.2$. The anisotropic force F_a is positive and the hydrostatic force F_h negative, since p_r increases outward toward zero. The Rastall force $F_\zeta = -(\zeta/\kappa)T'$ has the sign of ζ , because $T' < 0$: for $\zeta = -0.2$ it reinforces F_h against F_a , while for $\zeta = 0.2$ it acts with F_a , and the hydrostatic gradient must be steeper to compensate. In both cases the sum vanishes to within the accuracy of the numerical derivative, about 10^{-8} , which confirms that the solution obeys the modified conservation law (10).

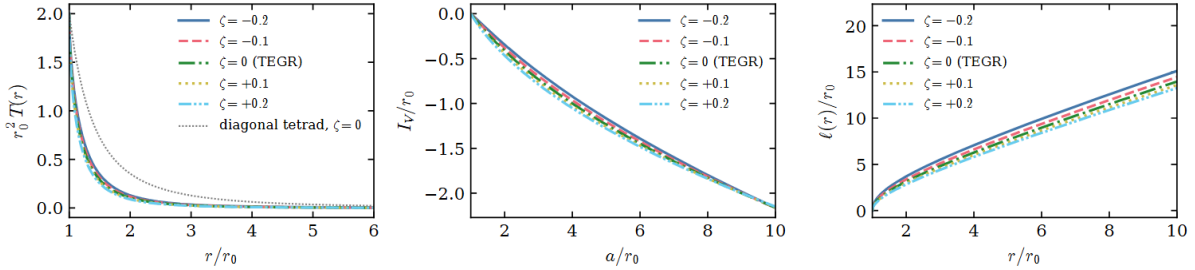


Figure 6: Left: torsion scalar for the non-diagonal tetrad and, for comparison, for the diagonal tetrad at $\zeta = 0$. Middle: volume-integral quantifier I_V as a function of the cut-off radius a . Right: proper radial distance $\ell(r)$ from the throat. All for $\omega = -2$.

5.4. Amount of exotic matter

The volume-integral quantifier [14] measures the total NEC-violating matter within a radius a . Using Eq. (22) and $b' - b/r = -(1-s)(\mathcal{A} + \mathcal{B}s)/q$ together with Eq. (28), the integrand collapses to a simple form:

$$\begin{aligned} I_V(a) &\equiv \oint (\rho + p_r) dV = \int_{r_0}^a \left(b' - \frac{b}{r} \right) dr \\ &= -2 \int_0^{s_a} r(s) s ds. \end{aligned} \quad (36)$$

In TEGR this integrates to $I_V = (1 + \omega) r_0 [(a/r_0)^{-1/\omega} - 1]$, which we used as a check of the numerics. I_V vanishes as $a \rightarrow r_0$, so the exotic matter can be made arbitrarily small by cutting the solution close to the throat. Figure 6 (middle) and Table 1 show that $|I_V|$ decreases as ζ becomes more negative: at $a = 2r_0$ the reduction from TEGR is about 17% for $\zeta = -0.2$, while $\zeta = 0.2$ increases it by about 13%. The effect weakens at large a , where all curves follow the common power law of Eq. (32). Because

$b \propto r^{-1/\omega}$ grows without bound, I_V diverges as $a \rightarrow \infty$ for this family, as for its GR limit; a finite total requires matching to an exterior. In Rastall teleparallel gravity with $\zeta \neq 0$ the Schwarzschild metric is not a vacuum solution with the non-diagonal tetrad, since its torsion scalar, does not vanish, so the exterior must be constructed within the theory. We leave this to future work.

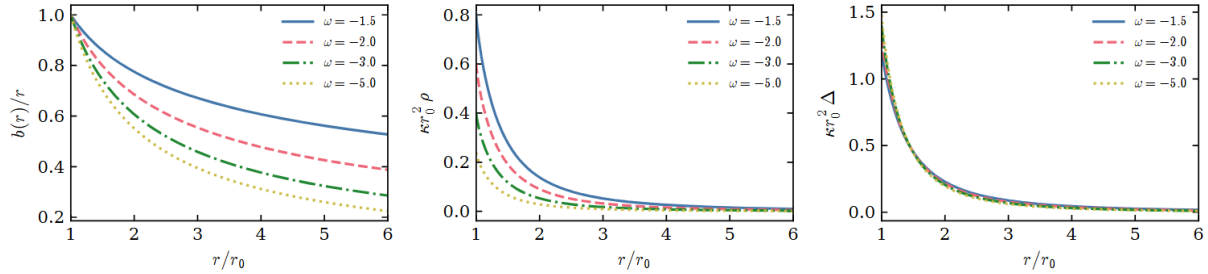


Figure 7: Dependence on the equation-of-state parameter at fixed $\zeta = 0.1$: b/r (left), energy density (middle) and anisotropy (right).

5.5. Dependence on the equation of state

Figure 7 fixes $\zeta = 0.1$ and varies ω . A more negative ω makes b/r fall faster, consistent with $b/r \propto r^{-(1+\omega)/\omega}$, and lowers the throat density $\kappa r_0^2 \rho_0 = (1 + 2\zeta)/|\omega|$. The anisotropy is almost independent of ω away from the throat. Near the phantom divide, $\omega \rightarrow -1^-$, the exponent $(1 + \omega)/\omega \rightarrow 0$, so b/r decays arbitrarily slowly and the throat region extends over many r_0 ; values of ω well below -1 give more compact wormholes.

5.6. Traversability

With $\Phi' = 0$ the radial tidal acceleration vanishes, and the lateral constraint of Morris and Thorne [4] for a traveller of size $|\eta| \simeq 2$ m reads $\gamma^2 v^2 |b' - b/r| |\eta| / (2r^2) \leq g_\oplus$. It is most restrictive at the throat, where $|b' - b/r| = (1 + 2\zeta)(1 + \omega)/\omega$, giving

$$\gamma v \leq r_0 \left[\frac{2g_\oplus \omega}{|\eta|(1 + 2\zeta)(1 + \omega)} \right]^{1/2}. \quad (37)$$

For $r_0 = 1$ km and $\omega = -2$ this allows non-relativistic crossings at a few km s^{-1} (Table 1), with negative ζ permitting faster passage because the throat is less sharply curved.

6. Conclusions

We have obtained an exact family of static traversable wormholes in Rastall teleparallel gravity with linear Rastall function, zero tidal force and the equation of state $p_r = \omega \rho$. The solution, Eqs. (29)–(30), is written in terms of $s = \sqrt{1 - b/r}$, reduces continuously to the power-law wormhole of GR at $\zeta = 0$, and satisfies the throat, flare-out and asymptotic-flatness conditions for $\omega < -1$ and $|\zeta| < 1/2$. The main conclusions are:

1. The tetrad choice has observable consequences. With the diagonal tetrad any non-zero Rastall coupling produces a solid-angle deficit or excess at infinity, whereas the non-diagonal tetrad keeps the solution asymptotically flat.
2. Because the Rastall term is isotropic, it drops out of $\rho + p_r$ and $\rho + p_t$. Violation of the radial NEC at the throat is therefore unavoidable for every ζ ; the coupling acts only through the shape function it selects.

3. The strong-energy combination equals $-2\zeta T/\kappa$ exactly, so negative coupling makes the SEC hold everywhere, a property the GR solution has only marginally.
4. Negative ζ reduces the volume-integral quantifier near the throat, relaxes the lateral tidal constraint, and lengthens the proper distance to a given radius. These gains come with a violation of the tangential DEC in a finite region around the throat for $\zeta \lesssim -0.1$.
5. The Rastall force $-(\zeta/\kappa)T'$ enters hydrostatic equilibrium on the same footing as the pressure gradient and the anisotropy, and the three balance exactly.

Natural extensions include a non-zero redshift function, non-linear $h(T)$ or $f(T)$, construction of an exterior vacuum solution of Rastall teleparallel gravity for matching, and dynamical stability under radial perturbations.

Conflict of Interest

The authors declare that there is no conflict of interest.

Data Availability

The data used in this study are available from the corresponding author upon reasonable request.

References

- [1] Flamm, L. (1916). Black Holes and Wormholes—The Physics of the Universe. *Phys. Z*, 17, 448.
- [2] Einstein, A., & Rosen, N. (1935). The particle problem in the general theory of relativity. *Physical Review*, 48(1), 73.
- [3] Misner, C. W., Thorne, K. S., & Zurek, W. H. (2009). John Wheeler, relativity, and quantum information. *Physics Today*, 62(4), 40-46.
- [4] Morris, M. S., & Thorne, K. S. (1988). Wormholes in spacetime and their use for interstellar travel: A tool for teaching general relativity. *American Journal of Physics*, 56(5), 395-412.
- [5] Morris, M. S., Thorne, K. S., & Yurtsever, U. (1988). Wormholes, time machines, and the weak energy condition. *Physical Review Letters*, 61(13), 1446.
- [6] Visser, M. (1995). Lorentzian wormholes. from einstein to hawking. *Woodbury*.
- [7] Hochberg, D., & Visser, M. (1998). Dynamic wormholes, antitrapped surfaces, and energy conditions. *Physical Review D*, 58(4), 044021.
- [8] Ellis, H. G. (1973). Ether flow through a drainhole: A particle model in general relativity. *Journal of Mathematical Physics*, 14(1), 104-118.
- [9] Bronnikov, K. A. (1973). Scalar-tensor theory and scalar charge. *Acta. Phys. Pol*, 4, B4.
- [10] Sushkov, S. (2005). Wormholes supported by a phantom energy. *Physical Review D—Particles, Fields, Gravitation, and Cosmology*, 71(4), 043520.
- [11] Lobo, F. S. (2005). Phantom energy traversable wormholes. *Physical Review D—Particles, Fields, Gravitation, and Cosmology*, 71(8), 084011.
- [12] Lobo, F. S. N., & Crawford, P. (2012). Classical Quantum Gravity 22, 4869 (2005); NM Garcia, FSN Lobo, and M. Visser. *Phys. Rev. D*, 86, 044026.

- [13] Poisson, E., & Visser, M. (1995). Thin-shell wormholes: Linearization stability. *Physical Review D*, 52(12), 7318.
- [14] Visser, M., Kar, S., & Dadhich, N. (2003). Traversable wormholes with arbitrarily small energy condition violations. *Physical review letters*, 90(20), 201102.
- [15] Lobo, F. S. N. (2009). Classical Quantum Gravity 25, 175006 (2008); FSN Lobo and MA Oliveira. *Phys. Rev. D*, 80, 104012.
- [16] Harko, T., & Lobo, F. S. (2011). S. i. Nojiri and SD Odintsov. *Phys. Rev. D*, 84, 024020.
- [17] Harko, T., Lobo, F. S., Mak, M. K., & Sushkov, S. V. (2013). Modified-gravity wormholes without exotic matter. *Physical Review D—Particles, Fields, Gravitation, and Cosmology*, 87(6), 067504.
- [18] Capozziello, S., Harko, T., Koivisto, T. S., Lobo, F. S., & Olmo, G. J. (2012). Wormholes supported by hybrid metric-Palatini gravity. *Physical Review D—Particles, Fields, Gravitation, and Cosmology*, 86(12), 127504.
- [19] Kleihaus, B., Kunz, J., Mojica, S., & Radu, E. (2016). Spinning black holes in Einstein–Gauss–Bonnet–dilaton theory: Nonperturbative solutions. *Physical Review D*, 93(4), 044047.
- [20] Rahaman, F., Kuhfittig, P. K. F., Ray, S., & Islam, N. (2014). Possible existence of wormholes in the galactic halo region. *The European Physical Journal C*, 74(2), 2750.
- [21] Aldrovandi, R., & Pereira, J. G. (2012). *Teleparallel gravity: an introduction*. Springer Science & Business Media.
- [22] J. W. Maluf, *Ann. Phys. (Berlin)* **525**, 339 (2013).
- [23] Bahamonde, S., Dialektopoulos, K. F., Escamilla-Rivera, C., Farrugia, G., Gakis, V., Hendry, M., & Di Valentino, E. (2023). Teleparallel gravity: from theory to cosmology. *Rep. Prog. Phys.*, 86(2), 026901.
- [24] Ferraro, R., & Fiorini, F. (2011). Spherically symmetric static spacetimes in vacuum $f(T)$ gravity. *Physical Review D—Particles, Fields, Gravitation, and Cosmology*, 84(8), 083518.
- [25] Bengochea, G. R., & Ferraro, R. (2009). Dark torsion as the cosmic speed-up. *Physical Review D—Particles, Fields, Gravitation, and Cosmology*, 79(12), 124019.
- [26] Linder, E. V. (2010). Einstein’s other gravity and the acceleration of the universe. *Physical Review D—Particles, Fields, Gravitation, and Cosmology*, 81(12), 127301.
- [27] Cai, Y. F., Capozziello, S., De Laurentis, M., & Saridakis, E. N. (2016). $f(T)$ teleparallel gravity and cosmology. *Reports on Progress in Physics*, 79(10), 106901.
- [28] Boehmer, C. G., Harko, T., & Lobo, F. S. (2012). Wormhole geometries in modified teleparallel gravity and the energy conditions. *Physical Review D—Particles, Fields, Gravitation, and Cosmology*, 85(4), 044033.
- [29] Jamil, M., Momeni, D., & Myrzakulov, R. (2013). Wormholes in a viable $f(T)$ gravity. *The European Physical Journal C*, 73(1), 2267.
- [30] Tamanini, N., & Boehmer, C. G. (2012). Good and bad tetrads in $f(T)$ gravity. *Physical Review D—Particles, Fields, Gravitation, and Cosmology*, 86(4), 044009.
- [31] CG, B. (2011). ohmer, A. Mussa and N. Tamanini. *Class. Quantum Grav*, 28, 245020.
- [32] Li, B., Sotiriou, T. P., & Barrow, J. D. (2011). $f(T)$ gravity and local Lorentz invariance. *Physical Review D—Particles, Fields, Gravitation, and Cosmology*, 83(6), 064035.

- [33] Krššák, M., & Saridakis, E. N. (2016). The covariant formulation of $f(T)$ gravity. *Classical and Quantum Gravity*, 33(11), 115009.
- [34] Golovnev, A., Koivisto, T., & Sandstad, M. (2017). On the covariance of teleparallel gravity theories. *Classical and quantum gravity*, 34(14), 145013.
- [35] Rastall, P. (1972). Generalization of the Einstein theory. *Physical Review D*, 6(12), 3357.
- [36] Rastall, P. (1976). A theory of gravity. *Canadian Journal of Physics*, 54(1), 66-75.
- [37] Visser, M. (2018). Rastall gravity is equivalent to Einstein gravity. *Physics Letters B*, 782, 83-86.
- [38] Darabi, F., Moradpour, H., Licata, I., Heydarzade, Y., & Corda, C. (2018). Einstein and Rastall theories of gravitation in comparison. *The European Physical Journal C*, 78(1), 25.
- [39] Darabi, F., Atazadeh, K., & Heydarzade, Y. (2018). Einstein static universe in the Rastall theory of gravity. *The European Physical Journal Plus*, 133(7), 249.
- [40] Hansraj, S., Banerjee, A., & Channuie, P. (2019). Impact of the Rastall parameter on perfect fluid spheres. *Annals of Physics*, 400, 320-345.
- [41] Oliveira, A. M., Velten, H. E. S., Fabris, J. C., & Casarini, L. (2015). Neutron stars in Rastall gravity. *Physical Review D*, 92(4), 044020.
- [42] Heydarzade, Y., & Darabi, F. (2017). Black hole solutions surrounded by perfect fluid in Rastall theory. *Physics Letters B*, 771, 365-373.
- [43] Moradpour, H., Heydarzade, Y., Darabi, F., & Salako, I. G. (2017). A generalization to the Rastall theory and cosmic eras. *The European Physical Journal C*, 77(4), 259.
- [44] Lin, K., & Qian, W. L. (2019). Neutral regular black hole solution in generalized Rastall gravity. *Chinese Physics C*, 43(8), 083106.
- [45] Bronnikov, K. A., Fabris, J. C., Piattella, O. F., & Santos, E. C. (2016). Static, spherically symmetric solutions with a scalar field in Rastall gravity. *General Relativity and Gravitation*, 48(12), 162.
- [46] Salako, I. G., Jawad, A., & Moradpour, H. (2018). Anisotropic compact stars in non-conservative theory of gravity. *International Journal of Geometric Methods in Modern Physics*, 15(06), 1850093.
- [47] Moradpour, H., Sadeghnezhad, N., & Hendi, S. H. (2017). Traversable asymptotically flat wormholes in Rastall gravity. *Canadian Journal of Physics*, 95(12), 1257-1266.
- [48] Halder, S., Bhattacharya, S., & Chakraborty, S. (2019). Wormhole solutions in Rastall gravity theory. *Modern Physics Letters A*, 34(12), 1950095.
- [49] Tangphati, T., Muniz, C. R., Pradhan, A., & Banerjee, A. (2023). Traversable wormholes in Rastall-Rainbow gravity. *Physics of the Dark Universe*, 42, 101364.
- [50] Saaïdi, K., & Nazavari, N. (2020). Traversable wormhole solutions in Rastall teleparallel gravity. *Physics of the Dark Universe*, 28, 100464.
- [51] Nazavari, N., Saaïdi, K., & Mohammadi, A. (2023). Wormhole solution in modified teleparallel-Rastall gravity and energy conditions: N. Nazavari et al. *General Relativity and Gravitation*, 55(3), 45.
- [52] Song, S. Q., & Güdekli, E. (2023). Traversable wormholes in Rastall teleparallel gravity with non-commutative geometry. *New Astronomy*, 100, 101993.

- [53] Ditta, A., Tiecheng, X., Errehymy, A., Mustafa, G., & Maurya, S. K. (2023). Anisotropic charged stellar models with modified Van der Waals EoS in $f(Q)$ gravity. *The European Physical Journal C*, 83(3), 254.
- [54] Ditta, A., Tiecheng, X., Mustafa, G., & Errehymy, A. (2023). The effect of modified hybrid and logarithmic teleparallel gravity on the interior solutions of compact stars. *The European Physical Journal C*, 83(11), 1020.
- [55] Ditta, A., Tiecheng, X., Mustafa, G., Sofuoğlu, D., & Mahmood, A. (2024). Dark energy compact stars in extended teleparallel gravity. *arXiv preprint arXiv:2401.06193*.
- [56] Ditta, A., Errehymy, A., Tiecheng, X., Mustafa, G., Alrebdi, H. I., & Abdel-Aty, A. H. (2022). Physical implications of teleparallel gravity on stellar configurations with modified Van der Waals equation of state: a broader view. *The European Physical Journal Plus*, 137(8), 933.
- [57] Daouda, M. H., Rodrigues, M. E., & Houndjo, M. J. S. (2012). Anisotropic fluid for a set of non-diagonal tetrads in $f(T)$ gravity. *Physics Letters B*, 715(1-3), 241-245.
- [58] Barriola, M., & Vilenkin, A. (1989). Gravitational field of a global monopole. *Physical Review Letters*, 63(4), 341.